Experiment 313: Estimation of the Gravitational Constant

Aims

The main aim of the experiment is to estimate the gravitational constant G. Secondary aims are to study torques, torsion balances, and damped harmonic oscillations. The experiment also provides an exercise and discussion on nonlinear curve-fitting, and error analysis.

Warning: Do not disturb the setup until you have read this handout. Mishandling of the apparatus can set it in motion, and if this happens you will need to wait hours for the motion to damp out.

References

[1] Richard Phillips Feynman, Robert B Leighton, and Matthew Sands. *The Feynman Lectures on Physics, Desktop Edition Volume I*, volume 1. Basic Books, 2013.

[2] George T Gillies. The Newtonian gravitational constant: recent measurements and related studies. *Reports on Progress in Physics*, 60(2):151, 1997.

[3] Peter J Mohr and Barry N Taylor. Codata recommended values of the fundamental physical constants: 2002. *Reviews of Modern Physics*, 77(1):1, 2005.

[4] Stephan Schlamminger. Fundamental constants: A cool way to measure big G. *Nature*, 510(7506):478–480, 2014.

[5] Roelof Karel Snieder and Kasper van Wijk. *A guided tour of mathematical physics.* Cambridge University Press, 3rd edition, 2015.

Introduction

Of the fundamental forces, the one of which we are most-directly aware is that of gravity. The force between two spherical masses m and M whose centre-to-centre separation is r is given by Newton's law of Universal Gravitation:

$$\mathbf{F} = -G\frac{mM\hat{\mathbf{r}}}{r^2}, \tag{1}$$

where G is the "gravitational constant," and the direction of $\hat{\mathbf{r}}$ is defined in Figure 1. Because gravity is by far the weakest of the fundamental forces, our estimate of G is one of the universal constants with the lowest precision.

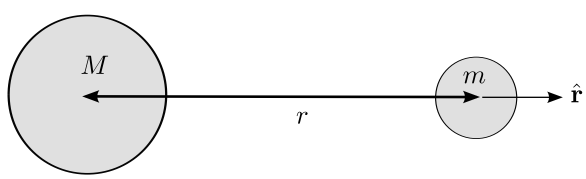

Figure 1: Two masses, separated by a distance r, and the direction of the unit vector $\hat{\mathbf{r}}$.

Following in the footprints of Henry Cavendish, we are going to estimate this constant G. The experiment you are about to perform is not very different from the way Cavendish did his in 1798.[1] Interestingly, our current "best" estimates of G are not that different from the Cavendish' result, either!

[1]Cavendish estimated the density of the Earth; his measurements included all the ingredients to estimate G, but he never did. Poynting did so, almost 100 years later, in 1894.

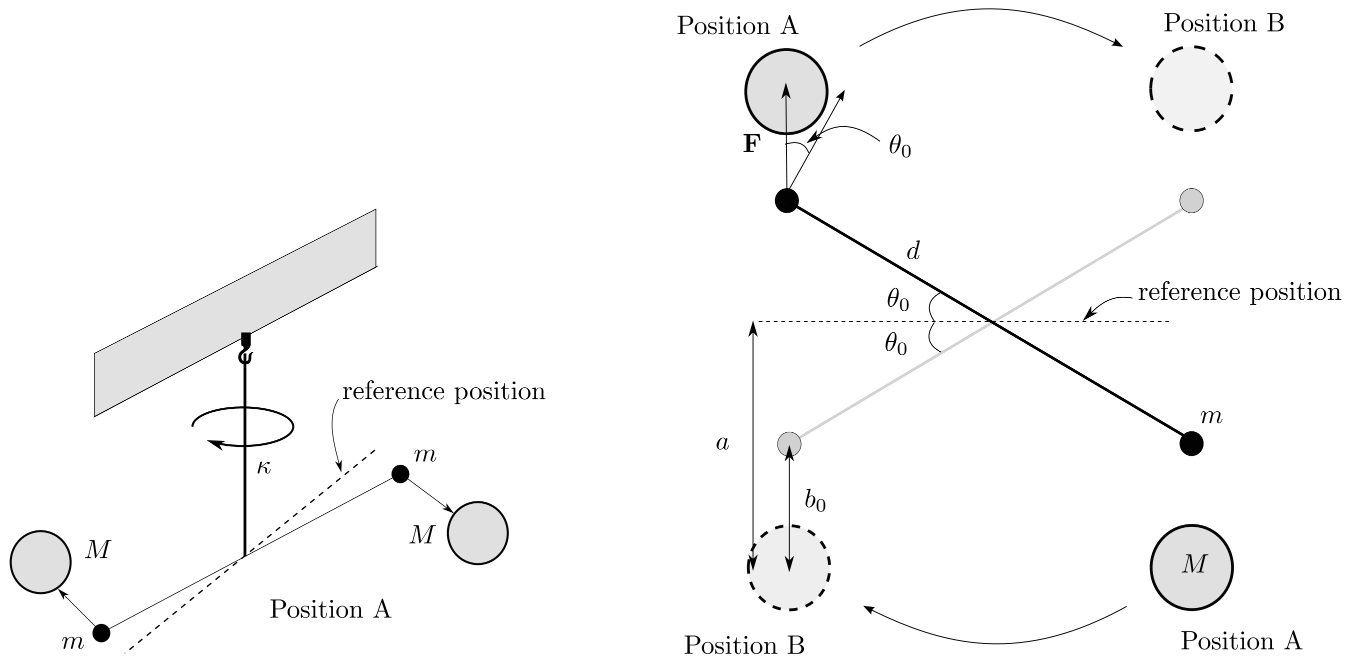

Figure 2: Left: Torsion balance with masses m, and torque constant κ, under the influence of two masses M. Right: Top view of the balance.

We will use a torsion balance, that is the modern-day equivalent of Cavendish' balance in 1798. It consists of a bar suspended by a very thin wire and carrying a small lead sphere of mass m at each end (Figure 2). To twist the balance an angle θ from its position at rest, requires a torque

$$\tau = \kappa\theta,$$

where κ is the torque constant, defined by the rigidity and dimensions of the wire.

Torque on the balance with M in Position A

The presence of a lead sphere of mass M, as shown in Figure 2, exerts a force $\mathbf{F}$ on m as defined by equation 1. With m rotating around an axis with a radius d, the force $\mathbf{F}$ results in a torque

$$\boldsymbol{\tau} = \mathbf{d} \times \mathbf{F}, \tag{2}$$

where $\times$ denoted the cross product, so that the direction of the vector $\boldsymbol{\tau}$ is according to the right-hand rule.

1. Use the right panel of Figure 2, to show that the torque on the torsion balance due to one mass M is $\left(\frac{GmM}{b_0^2}\cos\theta_0\right)d$.

With the balance at rest in position A at an angle θ_0 from the reference position, the gravitational attraction between the large and small spheres provides a torque that is exactly balanced by the restoring torque from the suspension wire:

$$2\left(\frac{GmM}{b_0^2}\cos\theta_0\right)d - \kappa\theta_0 = 0, \tag{3}$$

where b_0 is the distance between the spheres of mass m and M at rest.

2. Show that for small angles θ_0, $\cos\theta_0 \approx 1$, and that

$$G = \frac{\theta_0 b_0^2}{2mMd}\kappa. \tag{4}$$

If we know the torque constant κ, G is expressed in known or measured terms, only. The way to get κ experimentally is to oscillate the system, as in the next section.

Exciting damped harmonic oscillations: moving M to Position B

When the large spheres are rotated to position B, the suspended system will be deflected from its initial angular position $\theta = \theta_0$ and eventually reach a new rest position at the same angle on the opposite side of the reference position. However, the torsion balance will oscillate before equilibrium is restored. How many oscillations and their amplitude depend on the torque constant and friction in the wire and drag of the balance in the air, captured jointly in friction constant β.

The differential equation governing the oscillatory motion of the suspended system can be derived by applying Newton's second law of motion for a rotational system. The torque provided by the masses M is balanced by the rotational acceleration, a rotation velocity (with friction β) and the torque provided by the wire with constant κ:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}\theta}{\mathrm{d}t} + \kappa\theta = 2\frac{GmM}{b^2}d, \tag{5}$$

where the moment of inertia I [Chapter 18 in 1]. For the two discrete masses m, this is

$$I = \sum_i m_i d_i^2 = 2md^2. \tag{6}$$

For a bar of length $2d$ and ρ as its mass per unit length, the moment of inertia is:

$$I = \int_{-d}^{d} \rho x^2 dx. \tag{7}$$

3. Show that the total moment of inertia of rod and masses m is

$$I = 2md^2 + \frac{1}{12}m_{\mathrm{bar}}l_{\mathrm{bar}}^2 = 2md^2 + \frac{1}{12}m_{\mathrm{bar}}\left(2d\right)^2 = 2md^2\left(1 + \frac{1}{6}\frac{m_{\mathrm{bar}}}{m}\right). \tag{8}$$

The time-dependent forcing term

Now we are almost ready to solve the inhomogeneous differential equation of second order, as presented by Equation 5. The left-hand side is uniformly expressed in terms of $\theta(t)$, but the forcing term on the right-hand side is a function of mass separation b, and this separation is a function of time, as well.

4. Use Figure 2 to show that $b(t)$ can be expressed as

$$a - b(t) = d\sin(\theta(t)).$$

5. And use the small-angle approximation and a first order Taylor series approximation [e.g., Section 3.1 in 5] to show that

$$\frac{1}{b(t)^2} \approx \frac{1}{a^2}\left(1 + \frac{2d}{a}\theta(t)\right). \tag{9}$$

Equation (5) then becomes:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}\theta}{\mathrm{d}t} + K\theta = 2\frac{GmMd}{a^2}, \tag{10}$$

where

$$K = \kappa - \frac{4GmMd^2}{a^3} = \frac{2GmMd}{b_0^2\theta_0} - \frac{4GmMd^2}{a^3}. \tag{11}$$

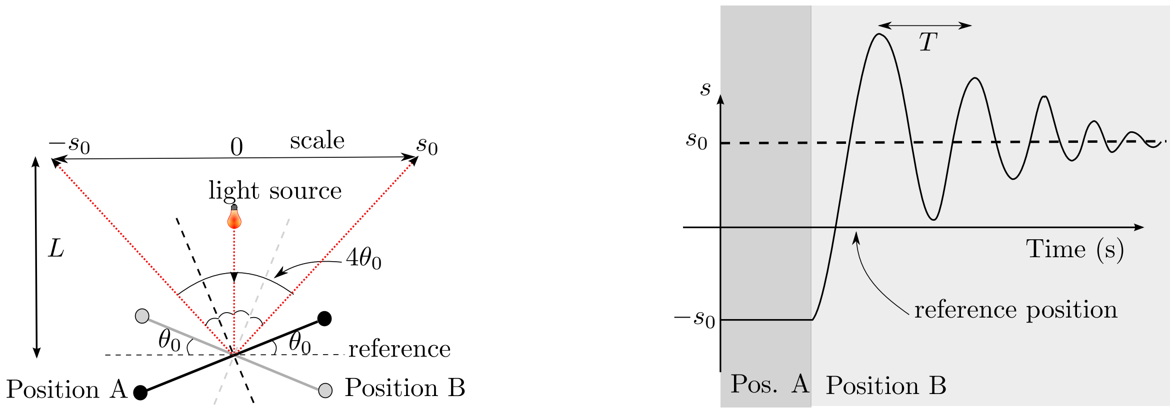

Figure 3: Left: Positions ($\pm s_0$) of the light beam at a distance L from the torsion balance. The light source is reflected from the mirror on the torsion balance under influence of the large masses M in Positions A and B. Right: Oscillations induced by moving the large masses M from Position A to Position B.

6. Rearrange equation (11), so that

$$G = \frac{b_0^2 \theta_0}{2mMd} \left(\frac{a^3}{a^3 - 2b_0^2 d\theta_0} \right) K. \tag{12}$$

All terms except K on the right-hand side are constants that we can measure in the lab, and K can be estimated indirectly by experiment as equation (10) is a standard second order differential equation which has a solution of the form:

$$\theta = A \exp\left(-\frac{\beta t}{2I}\right) \cos(\omega t + \phi) + \frac{2GMmd}{Ka^2}. \tag{13}$$

7. Show that expression 13 is a solution to the differential equation 10, where A and ϕ are constants and the angular frequency is

$$\omega = \sqrt{\frac{K}{I} - \frac{\beta^2}{4I^2}}, \tag{14}$$

Rearranging, we get

$$K = I\left(\omega^2 + \frac{\beta^2}{4I^2}\right). \tag{15}$$

The exponential decay of the oscillations

The oscillatory motion of the suspended system is studied by observing the source light reflected by the torsion balance onto the scale at distance L from the balance. The light spot moves from $s = -s_0$ and oscillates about $s = +s_0$ (Left panel of Figure 3). If L is large in the laboratory set-up), s may be treated as the arc subtended by the angle of deflection of the light beam which is twice the deflection of the suspended system (i.e. 2θ):

8. Use Figure 3 to write:

$$\theta_0 = \frac{2s_0}{4L} = \frac{s_0}{2L}, \tag{16}$$

so that

$$G = \frac{b_0^2 s_0}{4mMdL} \left(\frac{a^3}{a^3 - b_0^2 d s_0 / L} \right) \left(\omega^2 + \frac{\beta^2}{4I^2} \right) I. \tag{17}$$

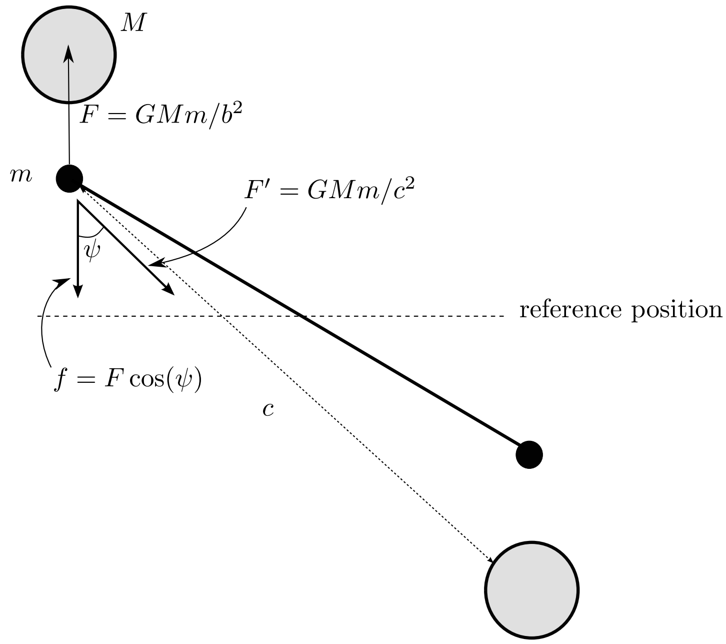

Figure 4: Forces F and F' on mass m due to both large spheres.

Thus s is proportional to θ and a plot of s against t (see the right panel of Figure 3), may be used to measure the quantities necessary for the determination of the gravitational constant G. From the plot of s against t, we can determine θ_0 and β/I, the former by measuring the initial and final position of the light spot ($-s_0$ and s_0), the latter by fitting the period and decay of the oscillations.

9. Insert equation (8), to get the following equation for the gravitational constant expressed in terms of measurable quantities:

$$G = \frac{\omega^2 b_0^2 d s_0}{2ML}\left[\frac{a^3}{a^3 - b_0^2 d s_0/L}\right]\left[1 + (\alpha/\omega)^2\right]\left[1 + \frac{1}{6}\frac{m_{\text{bar}}}{m}\right], \tag{18}$$

   where the damping parameter $\alpha = \beta/(2I)$.

Equation (18) has also been arranged so that the principal and secondary effects of the oscillatory motion of the suspended system are clear. All the factors in the square brackets represent secondary effects and contribute less than 10% to the value of G. From left to right, they account for (a) the varying gravitational torque, (b) effects of damping, and (c) consideration of the bar in the moment of inertia of the suspended system. However, the first term is not of the same form as the others, yet:

10. Use a first-order Taylor expansion [e.g., Section 3.1 in 5] to show that $\left[\frac{a^3}{a^3 - b_0^2 d s_0/L}\right] \approx [1 + b_0^2 d s_0/(a^3 L)]$, so that

$$G = \frac{\omega^2 b_0^2 d s_0}{2ML}\left[1 + \frac{b_0^2 d s_0}{a^3 L}\right]\left[1 + (\alpha/\omega)^2\right]\left[1 + \frac{1}{6}\frac{m_{\text{bar}}}{m}\right]. \tag{19}$$

Correction for the effect of the "distant" sphere

In the theory given above, the gravitational force exerted by the more distant of the two large spheres has not been taken into account. This additional force F' has a component f exactly opposite to the force F due to the closer sphere (Figure 4). Based on the geometry of Figure 4, we can express f in terms of F as follows:

$$f = F' \cos\psi = \frac{GmM}{c^2}\left(\frac{b_0 + 2d\sin\theta_0}{c}\right)$$

11. Show that $f = \gamma F$, with $F = GmM/b_0^2$ and

$$\gamma \approx \frac{b_0^3 + 2b_0^2 d\theta_0}{\left(b_0^2 + 4d^2 + 4b_0 d\theta_0\right)^{3/2}} = \frac{b_0^3 + b_0^2 ds_0/L}{\left(b_0^2 + 4d^2 + 2b_0 ds_0/L\right)^{3/2}}. \qquad (20)$$

To take into account the force f, we should multiply equation (4) by $[1-\gamma]$. As a result, equation (18) for the gravitational constant should be divided by $[1-\gamma]$.

12. Use a first-order Taylor expansion to show that $\left[\frac{1}{1-\gamma}\right] \approx [1+\gamma]$, and that our estimate of the gravitational constant is now

$$G = \frac{\omega^2 b_0^2 ds_0}{2ML}\left[1 + \frac{b_0^2 ds_0}{a^3 L}\right]\left[1 + (\alpha/\omega)^2\right]\left[1 + \frac{1}{6}\frac{m_{\text{bar}}}{m}\right][1+\gamma]. \qquad (21)$$

Correcting this systematic error, should improve your estimate of G, but even recent estimates of G must still suffer from systematic errors, as their quoted estimates plus uncertainties do not overlap [2, 3, 4].

Experiment

After you have answered all the previous questions:

13. Measure L, the distance from the mirror to the screen. From the measurements given below for the apparatus, determine a and b:

Data for the apparatus

Distance from the centre of the large mass M against the glass to the reference line (a)	4.22 ± 0.05 cm
Mass of large spheres (M)	1.500 ± 0.005 kg
Centre-to-centre distance between small spheres ($2d$)	10.00 ± 0.05 cm
Ratio of mass of bar to mass of small sphere (m_{bar}/m)	0.06 ± 0.01

Note that $a = 4.65 \pm 0.05$ cm for the PASCO unit.

14. Turn on the light source for the experiment to reflect on the scale
15. Measure the distance from wall to scale L
16. Set up the tablet with the tripod so that the light on the scale is in one end of the frame, and you have 20 cm to the other side.
17. On the tablet, select the lablet application, then camera, video, and set frame rate to 0.1 fps
18. Adjust light settings in the room so you can see the total scale bar (57 cm from end to end) in the video, and a clear light dot.
19. Start recording
20. After 5 minutes, flip the masses and record for two hours.
21. Stop recording (square button), select "done", and "save." This will result in a filename for your movie with the date in it.
22. Set the x/y coordinate system, set the scale bar, and calibrate knowing the frame is 57cm
23. Use the autopicking function on the lablet to track the light dot. Output goes to CSV file.
24. Connect tablet to a PC with USB, navigate to your folder, select the CSV file in your folder with the recording, and save this to the desktop PC

25. Just by **reading from a plot** of your data, you can estimate (by hand/eye), all the values that go in base value for G. If
$$G \approx \frac{\omega^2 b_0^2 d s_0}{2ML}$$
should get you the within an order of magnitude of G. If not, please check all your constants, units, etc.

26. **Fitting** the data to equation (13), estimate the initial and final steady-state levels of s: $\pm s_0$, the angular frequency ω, and the damping parameter $\alpha = \beta/(2I)$.

27. Estimate G from equation (19).

28. Estimate G from equation (21), which corrects for the attraction between m and the distant M.

29. Carry out an error analysis, including consideration of the approximations made in the formulae for determining the gravitational constant G.

List of Equipment

1. Gravitation torsion balance Leybold Model 332 10, or PASCO unit
2. Light source Leybold Model 450 60, or CW laser spot
3. A samsung tablet, with pin 6354(?)

May 2, 2018
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